Explicit generating series for small-step walks in the quarter plane

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Lattice walks are combinatorial objects that occur frequently in discrete mathematics, statistical physics, probability theory, or operational research. The generating series that enumerate them under certain constraints interest both combinatorialists and the algorithmists of computer algebra. First, their algebraic properties vary greatly according to the family of admissible steps chosen to define them, making their generating series sometimes rational, sometimes algebraic (and therefore described by a polynomial equation), sometimes D-finite (and therefore described by a linear differential equation), or sometimes with no apparent equation. Since a few years, this has motivated an effort of classification that has resulted in characterizations that are not sufficiently understood yet to be fully explicit. In addition, the computational properties of lattice walks make them an interesting challenge for computer algebra: indeed, their description often leads to equations, whether polynomial or differential, whose degrees, orders, and sizes are so large that it becomes difficult to obtain those descriptions explicitly, and to manipulate them with reasonable efficiency.

Given a family of non-zero vectors of the plane with coordinates ± 1, vectors which we shall call "steps", a small-step walk on the plane, square lattice is a finite succession of steps located one after the other. We are particularly interested in walks that are constrained by being confined to the quarter plane (that is, with non-negative integer coordinates), and counted according to their length (number of steps). In this talk, we shall present a work in progress that makes a bridge between previous works of different natures on the topic of small-step walks on the quarter plane. On the one hand, Bousquet-Mélou and Mishna showed [2] that among the 79 essentially different models of walks, only 19 possess a D-finite and transcendental generating series, and thus correspond to linear differential equations, but without making explicit the differential equations whose existence they were proving. Almost simultaneously, Bostan et Kauers [1] performed non-trivial but heuristic computations to obtain linear differential equations most probably satisfied by the 19 walk models, but without formally proving the correctness of these equations. In the work described, we shall give the first proof that these equations are satisfied by the corresponding generating series. Our approach proceeds by representing the generating series of constrained walks as coefficient extractions in rational series, and by a thorough validation of the use of the creative-telescoping process [3] employed for these extractions.

Once proved, the differential equations allow to compute in a guaranteed way many formulas and properties of the walk series. First, a suitable factorization of the underlying linear differential operators combined with the algorithm of [4] allows to represent the walks geenerating series as variations of iterated primitives of Gauss hypergeometric series. It follows that algebraicity and transcendence properties of enumerative series and specializations that are significant to combinatorics are accessible to computation, as well as asymptotic formulas for a number of walk models counted by lengths.

References

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