Definite Integration of Rational Functions

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Indefinite integration of rational functions has been a well-studied topic. Most computer algebra systems follow the text book algorithms, which first split the input rational function into a sum of polynomial part, rational part and transcendental part, and then integrate each of the three parts individually.

We will focus on the definite integration of the transcendental part, which is of the form g/f, where f and g are polynomials with coefficients in a field F of characteristic zero, f is monic, squarefree and non-constant, and g is nonzero and of lower degree than f. In particular, the talk will discuss how to evaluate limits of the antiderivative at $\pm\infty$ or at one of its discontinuities in closed form without the need to express the indefinite integral in terms of radicals.

The classical, straightforward formula for the indefinite integral of such a rational function is

$$\int \frac{g}{f} dx = \sum_{f(\alpha)=0} \frac{g(\alpha)}{f'(\alpha)} \ln(x-\alpha), \tag{1}$$

where the sum is over all roots α of f (which, by virtue of its squarefreeness, has only simple roots) and $g(\alpha)/f'(\alpha)$ is the residue of g/f at $x=\alpha$.

In the context of definite integration, say, over a real range $a..b$, a application of the Fundamental Theorem of Calculus (FTOC) to the antiderivative obtained from the classical formula may not be as straightforward as it sounds, due to branch cut issues. We illustrate this with a simple example.

$$F = \int \frac{dx}{x^2+1} = \sum_{\alpha^2=-1} \frac{\ln(x-\alpha)}{2\alpha} = \frac{i}{2}(\ln(x+i) - \ln(x-i)) \tag{2}$$

Note that, despite its appearance, F is actually real-valued. We apply FTOC to compute the improper integral from $-\infty..0$:

$$\int_{-\infty}^{0} \frac{dx}{x^2+1} = F(0) - \lim_{x\to\infty} F(-x) = -\frac{\pi}{2} - \frac{i}{2} \lim_{x\to\infty} (\ln(-x+i) - \ln(-x-i)) \tag{3}$$

The difficulty lies in evaluating the limit on the right. Each of the two logarithms is unbounded for $x \to \infty$, but the limit of the difference is finite. To see that, we rewrite each of the limits as follows:

$$\begin{aligned} \ln(i-x) &= \ln(x\cdot(-1+i/x)) = \ln(x) + \ln(-1+i/x), \\ \ln(-i-x) &= \ln(x\cdot(-1-i/x)) = \ln(x) + \ln(-1-i/x). \end{aligned}$$

The two $\ln(x)$ terms cancel, and for $x \to \infty$, we obtain

$$\begin{aligned}\lim_{x\to\infty}(\ln(-x+i)-\ln(-x-i)) &= \lim_{x\to\infty}\ln(-1+i/x)-\lim_{x\to\infty}\ln(-1-i/x)\\ &= \pi i-(-\pi i)=2\pi i\end{aligned}$$

Thus the definite integral (3) is evaluated as $\pi/2$, as expected.

In this derivation, we have used an explicit representation of the indefinite integral (2) in terms of radicals. However, it is well known from Galois theory that a representation in terms of radicals is not possible in general when the denominator polynomial is of degree 5 or higher, and even when it is, the corresponding radical expressions for the roots of f may become fairly unwieldy. Therefore it would be better if the limit could be computed using the implicit form (1).

Suppose again that our lower integration bound is $-\infty$, so we need to evaluate

$$\lim_{x\to\infty}\sum_{f(\alpha)=0}\frac{g(\alpha)}{f'(\alpha)}\ln(-x-\alpha). \tag{4}$$

Splitting the logarithm as $\ln(-x-\alpha)=\ln(x)+\ln(-1-\alpha/x)$ is valid in general when $x\to\infty$, but the limit for the second logarithm depends on the imaginary part of each particular root α:

$$\lim_{x\to\infty}\ln(-1-\alpha/x)=\begin{cases}\pi i & \text{if } \Im\alpha<0,\\ -\pi i & \text{if } \Im\alpha\ge 0\end{cases}$$

If f has only real roots, then the limit is $-\pi i$ for each such root, and we can find a closed form for the limit:

$$\begin{aligned}\lim_{x\to\infty}\sum_{f(\alpha)=0}\frac{g(\alpha)}{f'(\alpha)}\ln(-x-\alpha) &= (\ln(x)-\pi i)\lim_{x\to\infty}\sum_{f(\alpha)=0}\frac{g(\alpha)}{f'(\alpha)}\\ &= \begin{cases} 0 & \text{if } \deg g\le\deg f-2,\\ -\infty-\pi i & \text{if } \deg g=\deg f-1 \text{ and } \operatorname{lc}(g)>0,\\ \infty+\pi i & \text{if } \deg g=\deg f-1 \text{ and } \operatorname{lc}(g)<0,\end{cases}\end{aligned}$$

where $\operatorname{lc}(g)$ is the leading coefficient of g. In the general case, when f has non-real roots, it is not clear how to find a closed form for the limit without using radicals.

Besides the classical formula (1), other approaches to finding antiderivatives of rational functions with squarefree denominators have been proposed and widely implemented, with the goal of minimizing the size of the algebraic extension required to express the indefinite integral in closed form [2, 3, 4, 1]. These lead to antiderivatives of the form

$$\int\frac{g}{f}dx=\sum_{r(\beta)=0}\beta\ln s(\beta,x), \tag{5}$$

where r is a nonconstant univariate polynomial and s is a bivariate polynomial in β and x. Every root β of r is a residue $g(\alpha)/f'(\alpha)$, and the main advantage of these methods is that the minimal polynomial of β, which is r or a divisor of r, has smaller degree than f.

In this formulation, even though the degree of the algebraic extension is possibly lower, it is even less obvious how to find closed forms for the limit of the antiderivative, since in general s, the argument of the logarithm, is of degree 2 or higher.

Additional difficulties arise when the interval of integration contains poles of the integrand. In such a situation, the definite integral is undefined, but it can still be given a finite value as a Cauchy principal value integral. Then, in addition to the limits of the antiderivative at the integration bounds, limits at the poles of the integrand have to be computed.

The talk discusses new ways of evaluating such limits in closed form, by directly using one of the implicit representations (1) or (5), without the need to use radicals. Special cases such as when all roots of f are real, or when all roots are on the imaginary axis, can be recognized and lead to simpler formulas. In the general case, implicit representations for specific roots of f, such as, e.g., isolating intervals for real roots, are used.

References

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