Symbolic integration of multiple polylogarithms

E. Panzer

CNRS and IHES, Bures-sur-Yvette, France, erikpanzer@ihes.fr

Multiple polylogarithms are (locally) analytic, transcendental functions of several complex variables which generalize the logarithm $\mathrm{Li}_1(z) = -\log(1-z)$ to

$$\mathrm{Li}_{n_1,\ldots,n_d}(z_1,\ldots,z_d) = \sum_{0<k_1<\cdots<k_d} \frac{z_1^{k_1}\cdots z_d^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}} \quad \text{where} \quad n_1,\ldots,n_d \in \mathbb{N}. \tag{1}$$

They have many applications in mathematics but also in physics, where they occur frequently in computations of Feynman integrals. The algebra of rational linear combinations of such functions is in general not closed under integration, but there are special situations (subalgebras) where indeed the integral can itself be expressed in terms of multiple polylogarithms and their special values. For example,

$$\int_0^\infty\int_0^\infty \frac{\log(1+1/(xz))\,\mathrm{d}x\,\mathrm{d}y}{(1+y)(1+x+y+1/z)} = \zeta(3) - \frac{\pi^2}{6}\log(z) - \mathrm{Li}_{1,2}(1,-z) - \mathrm{Li}_3(-z) \tag{2}$$

with the Riemann zeta value $\zeta(3) = \sum_{k=1}^\infty k^{-3} = \mathrm{Li}_3(1)$. We sketch the algorithms from [1, 2] for symbolic computation of such integrals. These are based on representations in terms of *hyperlogarithms*, a family of iterated integrals of the form

$$L_{\sigma_1,\ldots,\sigma_n}(z) = \int_0^z \frac{\mathrm{d}z_1}{z_1-\sigma_1}\int_0^{z_1}\frac{\mathrm{d}z_2}{z_2-\sigma_2}\cdots\int_0^{z_{n-1}}\frac{\mathrm{d}z_n}{z_n-\sigma_n}. \tag{3}$$

The necessary condition of *linear reducibility* will be explained and we present our Maple implementation `HyperInt` [2], which has been applied successfully [3, 4] to compute many (including hitherto unknown) Feynman integrals. It may be useful in other applications as well, whenever integrals similar in spirit to Eq. (2) occur.

References

[1] F. C. S. Brown, *The Massless Higher-Loop Two-Point Function*, Commun. Math. Phys. **287**, pp. 925–958 (2009).

[2] E. Panzer, *Algorithms for the symbolic integration of hyperlogarithms with applications to Feynman integrals*, Comput. Phys. Commun. **188**, pp. 148–166 (2015).

[3] E. Panzer, *On hyperlogarithms and Feynman integrals with divergences and many scales*, JHEP **2014**, 71 (2014).

[4] E. Panzer, *On the analytic computation of massless propagators in dimensional regularization*, Nucl. Phys. B **874**, pp. 567–593 (2013).