Wave nature of particles

We have thus far developed a model of atomic structure based on the particle nature of matter:
Atoms have a dense nucleus of positive charge with electrons orbiting the nucleus in rather peculiar quantum orbits. Electrons are thus negatively charged \textit{particles}.
Or are they ?…. 
Wave nature of particles

**De Broglie wavelength**: De Broglie postulated that in analogy to light, matter could also have particle and wave characteristics.

Recall the energy, momentum relations for photons:

\[ E = hf \quad p = \frac{h}{\lambda} \]

Similarly a wavelength and frequency for matter waves can be defined as:

\[ f = \frac{E}{h} \quad \lambda = \frac{h}{p} \]
Wave nature of particles

De Broglie wavelength: De Broglie postulated that in analogy to light, matter could also have particle and wave characteristics.

Through what potential difference must an electron be accelerated to have a de Broglie wavelength of 500nm?

\[ p = mv = \frac{h}{\lambda} \Rightarrow v = \frac{h}{m\lambda} \]

\[ v = \frac{h}{m\lambda} = \frac{6.63 \times 10^{-34} \text{ J s}}{(9.11 \times 10^{-31} \text{ kg})(500 \times 10^{-9} \text{ m})} = 1456 \text{ m/s} \]

\[ e\Delta V = \frac{1}{2}mv^2 \Rightarrow \Delta V = \frac{mv^2}{2e} = \frac{(9.11 \times 10^{-31} \text{ kg})(1456 \text{ m/s})^2}{2(1.6 \times 10^{-19} \text{ C})} = 6.0 \times 10^{-6} \text{ V} \]
Wave nature of particles

**De Broglie wavelength:** De Broglie postulated that in analogy to light, matter could also have particle and wave characteristics.

A proton is confined to a nucleus of diameter 10 fm. What is its kinetic energy?

\[ p = mv = \frac{h}{\lambda} \Rightarrow v = \frac{h}{m\lambda} \]

\[ v = \frac{h}{m\lambda} = \frac{6.63 \times 10^{-34} \text{ J s}}{(1.67 \times 10^{-27} \text{ kg})(10 \times 10^{-15} \text{ m})} = 3.970 \times 10^{7} \text{ m/s} \]

\[ K = \frac{1}{2}mv^2 = \frac{1}{2}(1.67 \times 10^{-27} \text{ kg})(3.970 \times 10^{7} \text{ m/s})^2 \times \frac{1 \text{ eV}}{1.6 \times 10^{-19} \text{ J}} = 8.2 \text{ MeV} \]
De Broglie wavelength and the Bohr atom

De Broglie’s matter wave provided an explanation of the quantization of angular momentum in the Bohr atom:

Waves travelling in opposite directions in a confined space can set up a standing wave due to constructive interference.

A standing wave in a circle is formed when an integer number of wavelengths fits around the circumference:

\[ 2\pi r = n\lambda \]

Using the expression for the de Broglie wavelength we obtain the condition for quantized angular momentum:

\[
\lambda = \frac{h}{p} = \frac{h}{mv} \\
\Rightarrow mvr = n\frac{h}{2\pi} = n\hbar
\]
Davisson-Germer Experiment

Davisson and Germer showed that electron do behave like matter waves and can be diffracted.

Electrons are reflected from the surface of a nickel target. The surface layer of atoms in nickel acts as a **diffraction grating**.
**Davisson-Germer Experiment**

**Davisson and Germer** showed that electrons do behave like matter waves and can be diffracted.

Kinetic energy of incoming electrons:

\[
\frac{1}{2}mv^2 = eV \Rightarrow v = \sqrt{2Ve/m}
\]

Substituting into the expression for the de Broglie wavelength,

\[
\lambda = \frac{h}{mv} = \frac{h}{\sqrt{2Ve/m}}
\]

Given

\[
V = 54V \Rightarrow \lambda = \frac{h}{\sqrt{2Ve/m}} = 0.167\text{nm}
\]
Davisson and Germer showed that electron do behave like matter waves and can be diffracted.

Now consider the electrons hitting the surface of nickel:

After reflection from the surface, the electrons are observed to make a pattern of bright and dark spots. Can you explain why?
Davison-Germer Experiment

Davison and Germer showed that electron do behave like matter waves and can be diffracted.

Constructive interference occurs when the path difference between adjacent reflected electron waves is an integer number of wavelengths:

\[ d \sin \theta = n \lambda \]

Lattice spacing \( d \) in nickel is 0.215\( \text{nm} \). 1 st diffraction maximum was observed at 50 degrees:

\[ \lambda = \frac{d \sin \theta}{1} = 0.167\text{nm} \]

D-G experimental value for the electron wavelength agreed well with the de Broglie prediction.
Matter waves (but waves matter too)

- Electron diffraction was demonstrated by Davisson and Germer and G. P. Thomson (son of J. J. Thomson).

- Subsequently, diffraction for other matter waves such as atoms, neutrons and even molecules has been demonstrated.

- Atom interferometers can be constructed using standing waves of light as atomic mirrors and beam splitters.
Matter waves (but waves matter too)

- All matter can have wavelike properties.
- The **electron microscope** is based on the wave properties of electrons.

Check out [http://www.pbrc.hawaii.edu/microangela/](http://www.pbrc.hawaii.edu/microangela/) for more cool pictures
Matter waves

Why do we not observe matter wave effects for macroscopic objects?

De Broglie of a 1g object moving at 10m/s:

\[
\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{6.63 \times 10^{-34} \text{ Js}}{(1 \text{ g})(10 \text{ m/s})} = 6.63 \times 10^{-32} \text{ m}
\]

De Broglie wavelength of an electron moving at 0.02c:

\[
\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{0.02mec} = 0.36 \text{ nm}
\]

Macroscopic objects have wavelengths that are far too small to interact with slits etc to show interference or diffraction.
Matter waves

An electron (or any other microscopic particle) has a particular finite extent (size).

How do we describe a wave of a particular finite extent?
Wave packets

Single wave:
Consider a single wave which is a solution to the wave equation:

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{or} \quad \frac{k^2}{\omega^2} \frac{\partial^2 y}{\partial t^2}
\]

\[y = A \cos(kx - \omega t)\]

Wave number: \(k = \frac{2\pi}{\lambda}\)
Angular frequency: \(\omega = 2\pi f = \frac{2\pi}{T}\)

Phase velocity: \(v_p = \lambda f = \frac{\omega}{k}\)

Amplitude: A
Period: T

Such a wave has infinite extent and does not represent physical waves. All real waves are limited to finite regions: wave packets.
Wave packets

Superposition of waves:
Consider the superposition (addition) of two waves of equal amplitude but slightly different wavelengths and frequencies:

\[ y = y_1 + y_2 = A \cos(k_1 x - \omega_1 t) + A \cos(k_2 x - \omega_2 t) \]

\[ y(x, t) = 2A \cos \left( \frac{\Delta k x}{2} - \frac{\Delta \omega t}{2} \right) \cos(k x - \bar{\omega} t) \]

\[ \bar{k} = \frac{k_1 + k_2}{2}, \quad \bar{\omega} = \frac{\omega_1 + \omega_2}{2}, \]
\[ \Delta k = k_2 - k_1, \quad \Delta \omega = \omega_2 - \omega_1 \]

The new wave has a wave envelope modulating a high frequency wave.
Wave packets

Superposition of waves:

\[ y(x,t) = 2A \cos \left( \frac{\Delta k x}{2} - \frac{\Delta \omega t}{2} \right) \cos(\bar{k}x - \bar{\omega}t) \]

\[
\bar{k} = \frac{k_1 + k_2}{2}, \quad \bar{\omega} = \frac{\omega_1 + \omega_2}{2}, \quad \Delta k = k_2 - k_1, \quad \Delta \omega = \omega_2 - \omega_1
\]

The velocity of the envelope is **different** from the velocity of the wave within the envelope:

\[ v_p = \frac{\bar{\omega}}{\bar{k}} = \frac{\left(\omega_1 + \omega_2\right)}{2} \left(\frac{k_1 + k_2}{2}\right) \]

\[ v_g = \frac{\Delta \omega}{\Delta k} \]
Wave packets

Superposition of waves:

Adding two waves is not enough to create a wavepacket that is localized. One must add many waves to obtain a wave packet:

![Diagram of wave packets](Image)

Waves to be added span the frequency range from $f_0 - \frac{1}{2}\Delta f$ to $f_0 + \frac{1}{2}\Delta f$. The waves are all in phase at this instant of time.

The superposition of the many waves spanning a range of frequencies is a wave packet.

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Wave packets

Superposition of waves:

**Reciprocity:**
If the spatial extent of the envelope decreases then the range of wavelengths required increases.

If the temporal extent of the envelope decreases, then the range of frequencies required increases:

$$\Delta x \Delta k \approx 1$$
$$\Delta t \Delta \omega \approx 1$$
Wave packets

Superposition of waves:

Reciprocity:
A radio station broadcasts at 10.0 MHz. What range of frequencies is required to broadcast a 0.80µs pulse?

\[ \Delta t \Delta f \approx 1 \]

\[ \Rightarrow \Delta f \approx \frac{1}{\Delta t} = \frac{1}{0.8 \mu s} = 1.250 \text{ MHz} \]

Central frequency : 10.0 MHz

Frequency range : 10.0 MHz ± 0.6250 MHz

\[ \Rightarrow 9.375 \text{ MHz} < f < 10.625 \text{ MHz} \]
Wave packets

Superposition of waves:

**Phase velocity** of the wavepacket:

\[ v_p = \frac{\omega}{k} \]

**Group velocity**:

\[ v_g = \frac{d\omega}{dk} = \frac{d}{dk}(kv_p) \]

Taking the derivative, we get a **dispersion relation**:

\[ v_g = \frac{dv}{dk}(kv_p) = v_p + k \frac{dv_p}{dk} \]

\[ \frac{dv_p}{dk} = 0 \Rightarrow v_g = v_p \quad \text{(non dispersive medium)} \]

\[ \frac{dv_p}{dk} \neq 0 \Rightarrow v_g \neq v_p \quad \text{(wave packet spreading)} \]
Wave packets

Matter wave packets:

Individual matter waves have frequency and wavelength

\[ f = \frac{E}{h} \quad \lambda = \frac{h}{p} = \frac{2\pi}{k} \]

\[ \nu_p = f\lambda = \frac{E}{p} = \frac{\sqrt{\hbar^2 k^2 c^2 + m^2 c^4}}{\hbar k} = c\sqrt{1 + \frac{mc^2}{\hbar^2 k^2}} \geq c \]

The phase velocity of matter waves is greater than than or equal to the speed of light.

A wave packet made up of a range of wave numbers will show dispersion (spreading) even in free space.

\[ v_g = \frac{d}{dk} (k\nu_p) = \nu_p + k \frac{d\nu_p}{dk} = u \]

The group velocity of the matter wave packet is equal to the particle’s speed \( u \).
**Heisenberg Uncertainty Principle**

Minimum uncertainty:
\[
\Delta x \Delta p \geq \frac{\hbar}{2}, \quad \Delta \omega \Delta t \geq \frac{\hbar}{2}
\]

\[
p = \hbar k, \quad E = \hbar \omega
\]

- Our knowledge of conjugate quantities is inherently uncertain.
- No matter how good our measuring instruments are we cannot simultaneously know $x$ and $p$ or $E$ and $t$ with complete precision.

For Gaussian functions:
\[
\Delta p \Delta x = \frac{\hbar}{2}, \quad \Delta E \Delta t = \frac{\hbar}{2}
\]
Heisenberg Uncertainty Principle

Scientists can use lasers to trap sodium atoms in a box of length 1mm. What is the minimum temperature that the atoms can have?

\[ \Delta p \Delta x = m \Delta v \Delta x \geq \frac{\hbar}{2} \implies \Delta v \geq \frac{\hbar}{2m\Delta x} \]

Range of possible velocities centred around 0:

\[ -\frac{\hbar}{4m\Delta x} \geq v \geq \frac{\hbar}{4m\Delta x} \]

\[
T = \frac{mv_{\text{rms}}^2}{3k_B} = \frac{m}{3k_B} \left( \frac{1}{2} v_{\text{max}} \right)^2 = \frac{1}{3k_B} \left( \frac{\hbar^2}{64m\Delta x^2} \right)
\]

\[
= \frac{1}{3 \left( 1.38 \times 10^{-23} \text{ J/K} \right)} \left( \frac{\left( 1.05 \times 10^{-34} \text{ Js} \right)^2}{64 \left( 23 \times 1.67 \times 10^{-27} \text{ kg} \right) \left( 1 \times 10^{-3} \text{ m} \right)^2} \right) = 0.1 \times 10^{-15} \text{ K}
\]

Currently scientist have achieved cooling of about 1nK!