Energy-momentum relation (That Famous Equation…..)

Energy:

$$E = \gamma_u mc^2$$

Momentum:

$$p = \gamma_u mu$$

Squaring and subtracting the second equation from the first:

$$E^2 - p^2c^2 = m^2c^4\gamma^2{}_u\left(1 - \frac{u^2}{c^2}\right) = m^2c^4\gamma^2{}_u\frac{1}{\gamma^2{}_u} = m^2c^4$$

Energy-momentum relation:

$$\boxed{E^2 = p^2c^2 + \left(mc^2\right)^2}$$

If the particle is at rest, p=0. Thus the **rest energy** is

$$\boxed{E = mc^2}$$

Energy-momentum relation

$$E^2 = p^2c^2 + \left(mc^2\right)^2$$

Particles with zero mass (example: photons) have total energy:

$$E = pc$$

The rest mass of a particle **mc^2** is invariant in all inertial frames. Thus the quantity

$$E^2 - p^2c^2 = \left(mc^2\right)^2$$

is also invariant in all inertial frames.

Note: The total rest mass of a composite system is **not** equal to the sum of the rest masses of the individual particles.

Energy-momentum relation

$$E^2 = p^2c^2 + \left(mc^2\right)^2$$

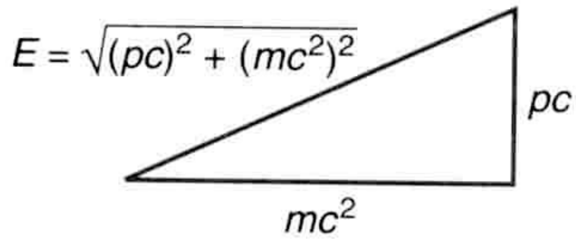
Energy is often expressed in **electron-volts (eV):**

$$1eV = 1.60 \times 10^{-19} J, \quad 1MeV = 10^6 eV$$
$$1J = 1kg\left(\frac{m^2}{s^2}\right)$$

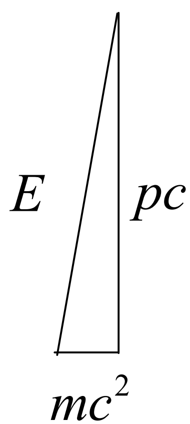
Some Rest Mass Values: Photon = 0 MeV, Electron = 0.511 MeV, Proton = 938.28 MeV

It is also convenient to express mass m and momentum p in energy units mc^2 and pc.

Energy-momentum relation

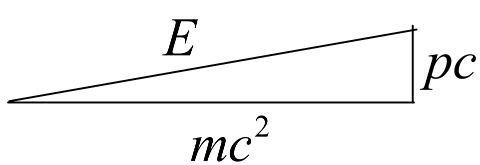


If u approaches c **(relativistic limit)**



$$E \approx pc$$

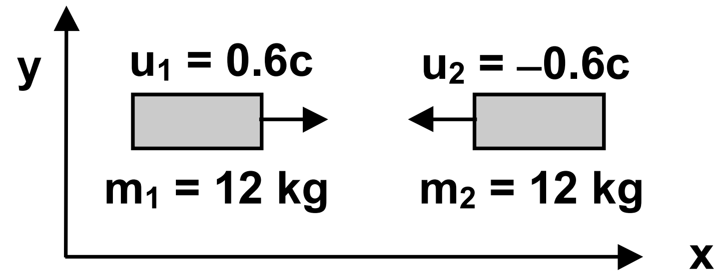
If u/c << 1 **(classical limit)**



$$E \approx mc^2$$

Solving for E, m, K and p

Two identical particles with rest mass 12 kg approach each other with equal but opposite velocities $u_1 = -u_2 = 0.6c$. Find the total rest mass of this system.



$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}} = \frac{1}{\sqrt{1-(0.6)^2}} = 1.25$$

$$\boxed{E_{tot}} = E_1 + E_2 = 2\gamma mc^2 = 2(1.25)(12\ kg)c^2 = \boxed{30\ kg \cdot c^2}; \qquad \boxed{p_{tot} = 0}$$

$$\boxed{m_{tot}} = \sqrt{\left(\frac{E_{tot}}{c^2}\right)^2 - \left(\frac{p_{tot}}{c}\right)^2} = \sqrt{\left(\frac{30\ kg \cdot c^2}{c^2}\right)^2 - 0} = \boxed{30\ kg}$$

Solving for E, m, K and p

Find the kinetic energy, rest energy and total energy of a 1g particle with speed of 0.8c.

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-(0.80)^2}} = \frac{5}{3} \Rightarrow \gamma_{\mathbf{u}} - 1 = \frac{5}{3} - 1 = \frac{2}{3}$$

$$E_0 = mc^2 = \left(1.0 \times 10^{-3} \text{ kg}\right)\left(3.0 \times 10^8 \text{ m/s}\right)^2 = 9.0 \times 10^{13} \text{ J}$$

The rest energy is $E_0 = mc^2 = 9.0 \times 10^{13}$ J.

The kinetic energy is $K = \left(\gamma_{\mathrm{p}} - 1\right)E_0$

$$\Rightarrow K = \left(\frac{2}{3} - 1\right)\left(9.0 \times 10^{13} \text{ J}\right) = 3.0 \times 10^{13} \text{ J}$$

The total energy is $E = E_0 + K = 9.0 \times 10^{13}\text{ J} + 3.0 \times 10^{13}\text{ J} = 1.2 \times 10^{14}$ J.

Solving for E, m, K and p

At what speed is a particle's kinetic energy twice its rest energy?

$E = \gamma_u mc^2 = E_0 + K$. For $K = 2E_0$,

$$\gamma_u mc^2 = E_0 + 2E_0 = 3mc^2$$

$$\Rightarrow \gamma_\mathbf{u} = \frac{1}{\sqrt{1 - u^2/c^2}} = 3 \Rightarrow 1 - \frac{u^2}{c^2} = \frac{1}{9} \Rightarrow u = \frac{\sqrt{8}}{3}c = \mathbf{0.943}c$$

Solving for E, m, K and p

What is the speed and momentum pc (MeV) of an electron (rest mass 0.511 MeV) with a total energy of 12 MeV?

$$\gamma_u = \frac{E}{mc^2} = \frac{12\ \mathrm{MeV}}{0.511\ \mathrm{MeV}} = 23.483$$

$$\frac{u}{c} = \sqrt{1-\frac{1}{\gamma_u{}^2}} = \sqrt{1-\frac{1}{(23.48)^2}} = \quad 0.9991$$

$$pc = \sqrt{E^2 - \left(\mathrm{mc}^2\right)^2} = \sqrt{(12\ \mathrm{MeV})^2 - (0.511\ \mathrm{MeV})^2} = 11.99\,MeV$$

Conservation of mass-energy

The sum of the relativistic total energies of a system of particles before interaction is equal to the sum of the relativistic total energies of the system of particles after the interaction. The total relativistic energy of the i_{th} particle in the system is given by

$$E_i = \gamma_{u_i} m_i c^2 = \frac{m_i c^2}{\sqrt{1 - u_i^2 / c^2}}$$

Conservation of mass-energy

Consider an inelastic collision of two particles of equal mass approaching each other with speed v in the lab frame S.

Applying the conservation of mass-energy:

$$\gamma_{u_1} mc^2 + \gamma_{u_2} mc^2 = \gamma_{u_f} Mc^2$$

$$M = \frac{2m}{\sqrt{1 - u^2/c^2}} = 2m + \frac{2K}{c^2}$$

Note that **total mass is not conserved**. Kinetic energy has been converted to mass.

Conservation of mass-energy

Nuclear Fission: The decay of a heavy nucleus into several lighter nuclei.
Example:

$$^{236}U \rightarrow {}^{144}Ba + {}^{89}Kr + 3n$$

By conservation of mass-energy:

$$m_U c^2 = \frac{m_{Ba} c^2}{\sqrt{1 - u_{Ba}^2 / c^2}} + \frac{m_{Kr} c^2}{\sqrt{1 - u_{Kr}^2 / c^2}} + \frac{m_{n_1} c^2}{\sqrt{1 - u_{n_1}^2 / c^2}} + \frac{m_{n_2} c^2}{\sqrt{1 - u_{n_2}^2 / c^2}} + \frac{m_{n_3} c^2}{\sqrt{1 - u_{n_3}^2 / c^2}}$$

The rest mass of the uranium nucleus is **greater** than the sum of the masses of the daughter products. The extra mass is converted to kinetic energy causing the products of the fission to be ejected at extremely high velocities.

Nuclear fission is the basis of nuclear reactors as well as nuclear bombs.

Fusion: Collision of nucleii in which the kinetic energy is used to overcome the repulsion of protons to form a heavier nucleus. In this case the total incoming energy must equal the rest energy + the binding energy:

Conservation of mass-energy

A Σ particle decays into a neutron (pc = 4702 MeV) and pion (pc = 169 MeV). Find the total rest mass and kinetic energy of the Σ particle.

$$\boxed{E_n} = \sqrt{(pc)_n^{\ 2} + \left(mc^2\right)_n^{\ 2}} = \sqrt{(4702\text{ MeV})^2 + (940\text{ MeV})^2} = \boxed{4795\text{ MeV}}$$

$$\boxed{E_\pi} = \sqrt{(pc)_\pi^{\ 2} + \left(mc^2\right)_\pi^{\ 2}} = \sqrt{(169\text{ MeV})^2 + (140\text{ MeV})^2} = \boxed{219\text{ MeV}}$$

$$\boxed{E_\Sigma} = E_n + E_\pi = 4795\text{ MeV} + 219\text{ MeV} = \boxed{5014\text{ MeV}}$$

$$\boxed{\left(mc^2\right)_\Sigma} = \sqrt{E_\Sigma^{\ 2} - (pc_\Sigma)^2} = \sqrt{(5014\text{ MeV})^2 - (4871\text{ MeV})^2} = \boxed{1189\text{ MeV}}$$

where $pc_\Sigma = pc_n + pc_\pi = 4702\text{ MeV} + 169\text{ MeV} = 4871\text{ MeV}$

$$\boxed{K_\Sigma} = E_\Sigma - \left(mc^2\right)_\Sigma = 5014\text{ MeV} - 1189\text{ MeV} = \boxed{3825\text{ MeV}}$$

Conservation of mass-energy

The reaction that powers the sun consists of the fusion of four protons into a helium nucleus. How much energy is released in each fusion? (Mass of a helium nucleus = 6.64 x 10^{-27}kg).

Mass lost in the fusion process:

$$4m_\mathrm{p} - m_\mathrm{He} = 4(1.67 \times 10^{-27}\ \mathrm{kg}) - 6.64 \times 10^{-27}\ \mathrm{kg} = 0.04 \times 10^{-27}\ \mathrm{kg}$$

Energy released:

$$E = mc^2 = (0.04 \times 10^{-27}\ \mathrm{kg})\ (3.0 \times 10^8\ \mathrm{m/s})^2 = 3.6 \times 10^{-12}\ \ \mathrm{J}$$

General Relativity

Postulates:
The laws of physics are the same in all frames (including accelerating frames).

Principle of equivalence: A gravitational field in a region around a point is equivalent to an accelerating reference frame.

Consequences:
- Curvature of spacetime
- Explanation of gravity
- Gravitational time dilation, gravitational red shift